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Surface Currents on a Conducting Sphere Excited by a Dipole*

C. H. Papas and Ronold King
Cruft Laboratory, Harvard University, Cambridge, Massachusetts

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This paper treats the problem of determining the current distribution on the surface of a perfectly conducting sphere when driven by a dipole antenna erected on its surface. Curves of the real and imaginary parts of the surface currents are given for the case of a half-wave dipole and various radii of the sphere.

I. INTRODUCTION

The distribution of currents on the surface of a metal spheroid excited by an adjacent antenna or by a cavity is of great practical interest (see Fig. 1). The analytical problem involved is closely related to the determination of the diffracted field of a plane wave incident upon a sphere or spheroid,[1] which has been studied by many investigators. The rigorous solution of the general case in which the wavelength of the incident plane wave is of the same order of magnitude as the dimension of the diffracting sphere is due to L. Brillouin. In practice a plane wave is approximated by the field of a radiator situated a great many wavelengths from the diffracting sphere or prolate spheroid. This field induces surface currents which, in turn, set up an electromagnetic field. This induced field is so disposed that on the surface of the sphere or spheroid (assumed perfectly conducting) the sum of the tangential components of the incident and induced E-fields vanishes and the sum of the tangential components of the H-fields equals the surface current density. Since the scalar wave equation is separable in spherical (and spheroidal) coordinates, these boundary conditions permit the determination of the unknown constants of the infinite series solution for the induced E- and H-fields.

If the distant radiator is brought close to the diffracting body, the incident field is no longer a plane wave and the analytical problem becomes so complicated that the general procedure described above has to be abandoned. An alternative procedure has been formulated by Feld[2] who has applied Lorentz' lemma to a linear antenna over a perfectly conducting sphere. The current in the antenna is assumed to be sinusoidal; the coefficients of the modal currents on the sphere are determined by the well-known Fourier-Lamé method.[3]

By applying the general reciprocity theorem (Appendix A) to a closed metallic surface with a linear antenna erected upon it, an integral equation is obtained. In order to solve this integral equation by the Fourier-Lamé method it is necessary to restrict the closed metallic surface to a shape that may be described simply using coordinates in which the homogeneous scalar wave equation,

$$\nabla^2\psi + k^2\psi = 0,$$

is separable.[4] Another limitation (which is of purely practical importance) is the availability of numerical tables of the functions used in the solutions. For example, if the closed metallic surface is a sphere, the "natural" coordinates are spherical which permit separation of variables and give as angular solutions the Legendre polynomials and as radial solutions the spherical Hankel functions. Both the Legendre polynomials[5] and spherical Hankel functions[6] have been tabulated. Similarly, if the metallic surface is a prolate spheroid, the prolate spheroidal coordinates[1] ξ, η, ϕ are used. These also permit

* The research reported in this document was made possible through support extended Cruft Laboratory, Harvard University, jointly by the Navy Department (Office of Naval Research) and the Signal Corps, U. S. Army under Contract N5 ori-76, T. O. I.

[1] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

[2] J. N. Feld, Doklady U.S.S.R. **51**, No. 3, 203–206 (1946).

[3] G. Grunberg, J. Phys. U.S.S.R. **10**, No. 4, 301–320 (1946).

[4] L. P. Eisenhart, Annals Math. **35**, No. 2, 284–305 (1934).

[5] *Tables of Associated Legendre Functions* (Columbia University Press, New York, 1945).

[6] *Tables of Spherical Bessel Functions* (Columbia University Press, New York, 1945).

separation of variables and yield the functions $Se_l^1(\eta)$ as "angular" solutions and $Re_l^1(\xi)$ as "radial" solutions. These functions have been tabulated.[7] It is evidently possible to solve for the surface currents on spheres, prolate spheroids, and oblate spheroids, and from these to determine the transmitting and receiving characteristics of the array consisting of linear antenna and closed surface.

Although the spheroidal functions have been tabulated, the additional computation necessary to determine the surface currents is considerable. Therefore, the simpler problem of a linear antenna over a sphere has been carried through first.

II. LINEAR ANTENNA OVER SPHERE

For an array consisting of a linear antenna of height h over a perfectly conducting sphere of radius a, the total current crossing a parallel of latitude θ on the sphere is (Appendix C)

$$I_{1S}(\theta)=\sum_{n=0}^{\infty} B_n P_n(\cos\theta), \tag{1}$$

where

$$\begin{aligned}B_n=(I_{\max}/2)\{&[\rho_{n-1}(kd)/\rho_{n-1}'(ka)]\\ &-\cos kh[\rho_{n-1}(ka)/\rho_{n-1}'(ka)]\\ &-[\rho_{n+1}(kd)/\rho_{n+1}'(ka)]\\ &+\cos kh[\rho_{n+1}(ka)/\rho_{n+1}'(ka)]\},\end{aligned} \tag{2}$$

and $k=2\pi/\lambda$, a is the radius of sphere, h is the height of antenna, $d=a+h$. It was assumed in the derivation of (1) that the current along the antenna has the form

$$I_{1A}(r)=I_{\max}\sin k(d-r),$$

and that this distribution is independent of currents on the sphere. For very thin antennas driven at the junction with the sphere a sinusoidal distribution is a good approximation for lengths differing considerably from $\lambda/2$. In general, a sinusoidal current cannot be maintained by a single generator but requires a distribution of generators along the entire antenna.

When the length of the antenna is $\lambda/4$, the current is maximum, $I_{\max}$, at the base $(r=a)$ and zero at the top $(r=a+h)$. Since $kh=\pi/2$, the expression for B_n reduces to

$$\begin{aligned}B_n=(I_{\max}/2)\{&[\rho_{n-1}(kd)/\rho_{n-1}'(ka)]\\ &-[\rho_{n+1}(kd)/\rho_{n+1}'(ka)]\}.\end{aligned} \tag{3}$$

Calculations have been made for various values of the radius a with $kh=\pi/2$ using (3). In order to compute terms such as the ratio

[7] Stratton, Morse, Chu, and Hutner, *Elliptic Cylinder and Spheroidal Wave Functions* (Massachusetts Institute of Technology Press, Cambridge, 1941).

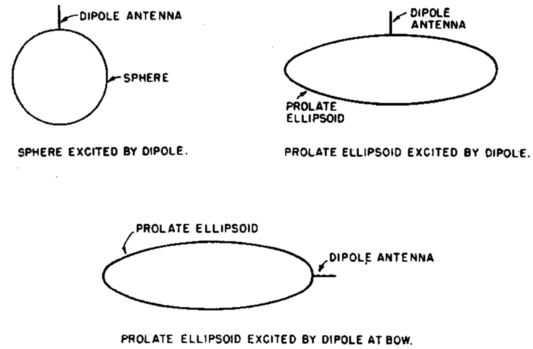

FIG. 1. Some closed metallic surfaces excited by dipoles which may be solved for surface currents by the method of this paper.

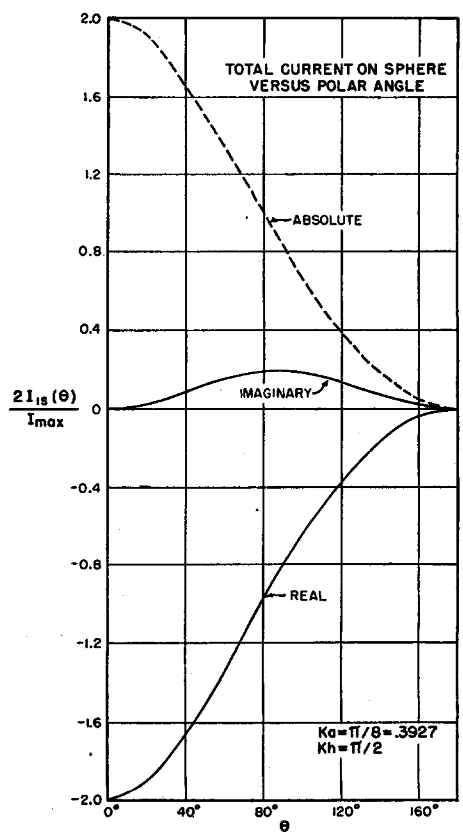

FIG. 2. Total current on sphere *versus* polar angle for $ka=\pi/8$.

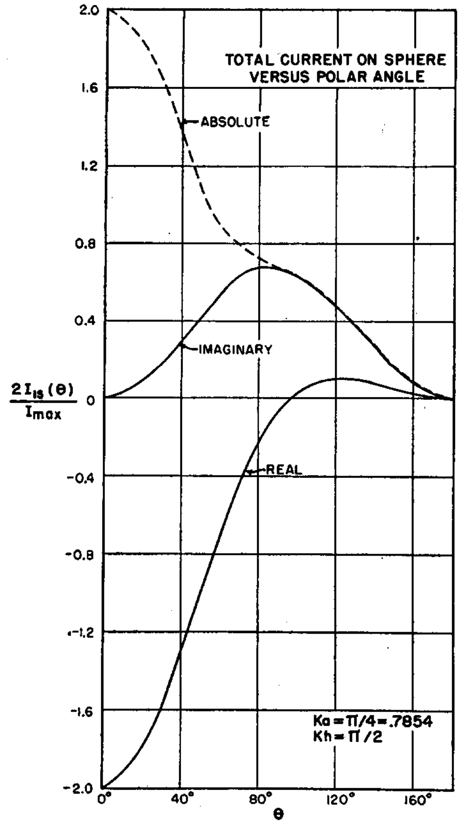

FIG. 3. Total current on sphere *versus* polar angle for $ka=\pi/4$.

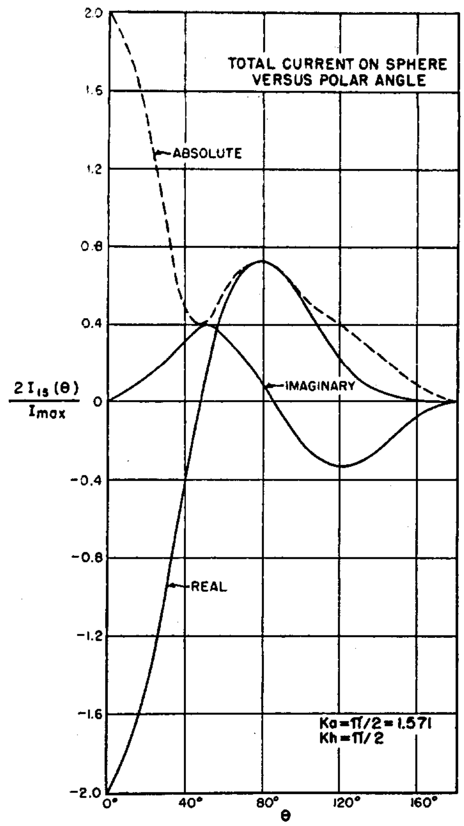

FIG. 4. Total current on sphere *versus* polar angle for $ka=\pi/2$.

$\rho_{n-1}(kd)/\rho_{n-1}'(ka)$ it was found convenient to follow a computational procedure used by L. Infeld.[8] By factoring this ratio it is seen that one of the factors is a function which satisfies Riccati's equation and for which there is a recurrence relation. That is,

$$\rho_{n-1}(kd)/\rho_{n-1}'(ka)=\rho_{n-1}(ka)/\rho_{n-1}'(ka)\cdot\rho_{n-1}(ka+kh)/\rho_{n-1}(ka)=\zeta_{n-1}(ka)\cdot\rho_{n-1}(ka+kh)/\rho_{n-1}(ka), \quad (4)$$

where

$$\zeta_{n-1}(ka)=\rho_{n-1}(ka)/\rho_{n-1}'(ka).$$

This zeta satisfies Riccati's equation and equals the zeta used by Infeld with the n equal to his $2m+1$. Letting $x=ka$, the recurrence formula is

$$\zeta_n(x)=[nx\zeta_{n-1}(x)-x^2]/[(x^2-n^2)\zeta_{n-1}(x)+nx], \quad (5)$$

and $\zeta_{-1}(x)=+i$. Starting with $\zeta_{-1}(x)=+i$, the entire set of functions $\zeta_0(x)$, $\zeta_1(x)$, $\cdots$ was obtained.

The other factor which is of the form $\rho_{n-1}(ka+kh)/\rho_{n-1}(ka)$ was computed from the fundamental definition of the functions:

$$\frac{\rho_{n-1}(ka+kh)}{\rho_{n-1}(ka)}=\frac{(x+kh)[(\pi/2(x+kh))^{\frac{1}{2}}J_{n+\frac{1}{2}}(x+kh)+(i/\cos n\pi)(\pi/2(x+kh))^{\frac{1}{2}}J_{-(n+\frac{1}{2})}(x+kh)]}{x[(\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x)+i(1/\cos n\pi)(\pi/2x)^{\frac{1}{2}}J_{-(n+\frac{1}{2})}(x)]}. \quad (6)$$

The functions

$$(\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x) \quad \text{and} \quad (\pi/2x)^{\frac{1}{2}}J_{-(n+\frac{1}{2})}(x)$$

are the spherical Bessel functions, tables of which are available.

[8] L. Infeld, Q. App. Math. 5, No. 2, 113–132 (1947)

Using (6) and (5), the ratios $\rho_{n-1}(ka+kh)/\rho_{n-1}'(ka)$ were computed for $n=0, 1, 2, 3, 4\cdots 11$ when $kh=\pi/2$ and $ka=\pi/8$, $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$, $7\pi/4$, and 2π. These ratios were substituted into (3). The values of B_n thus obtained were substituted into (2) and $I_{1S}(\theta)$ was determined.

III. RESULTS

The real and imaginary parts of the current $I_{1S}(\theta)$ crossing the θ parallel of latitude have been computed for several values of the radius a of the perfectly conducting sphere in the special case of an antenna of length $h=\lambda/4$ erected on the sphere. Plots of these currents are shown in Figs. 2–10. The ordinate of these plots is twice the ratio of $I(\theta)$ to $I_{\max}$. Since the positive direction of current along the antenna is towards its top and the positive direction of flow on the sphere is from its North pole towards its South, it is necessary for $I_{1S}(0)$ to be opposite in sign to $I_{1A}(a)=I_{\max}$. In all the plots for $\theta=0$ the imaginary part is zero and the real part equals -2. That is,

$$2I_{1S}(0)/I_{\max}=-2 \quad \text{or} \quad I_{1S}(0)=-I_{\max}.$$

At the bottom of the sphere, $\theta=180°$, the current is zero.

In the theoretical derivation it was assumed that there was a generator or a distribution of generators somewhere in the antenna. The only restriction upon this generator is that it be independent of the angle ϕ. For example, the generator may be a single idealized slice generator or a distribution of these along the antenna. In the computed case the current distribution is assumed to be sinusoidal with a node at the top of the antenna and a loop $I_{\max}$ at its base. This requires a single slice generator to be placed at the base of a very thin antenna as indicated in Fig. 11. The current distributions shown in the plots are for just such a model.

Since an actual physical antenna of finite cross section driven from a coaxial line over an imperfectly conducting sphere necessarily differs from the ideal model assumed in the theory, actual currents must be expected to differ somewhat from the computed currents represented in the plots. However, it is safe to assume that the currents in the idealized mathematical model are

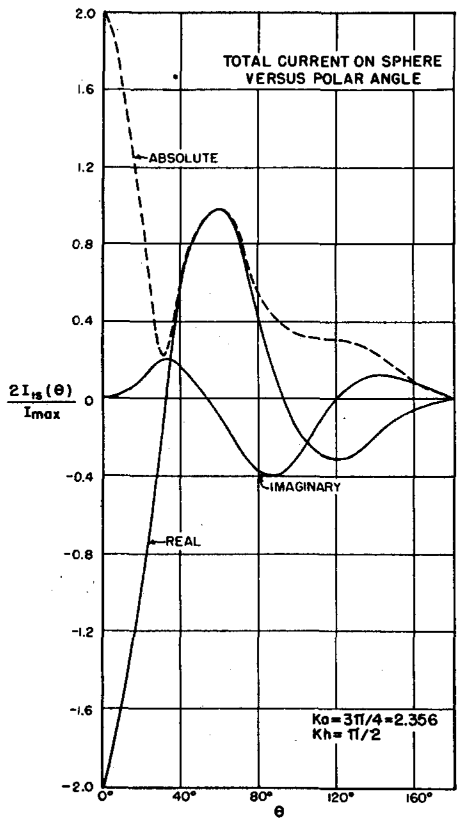

FIG. 5. Total current on sphere *versus* polar angle for $ka=3\pi/4$.

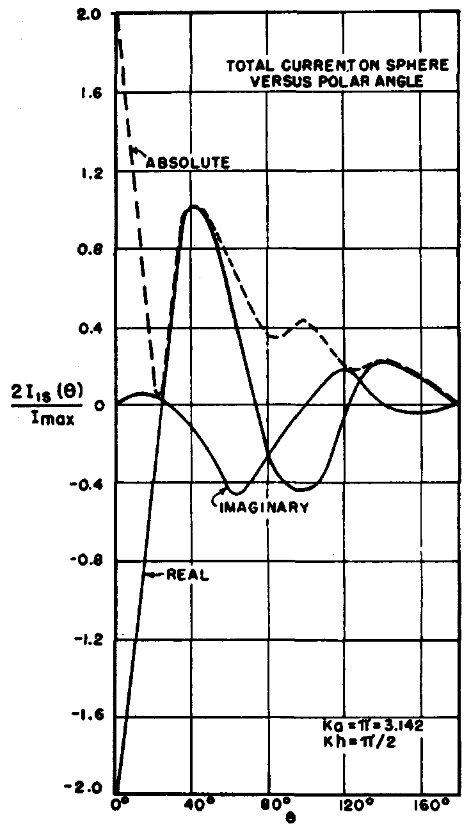

FIG. 6. Total current on sphere *versus* polar angle for $ka=\pi$.

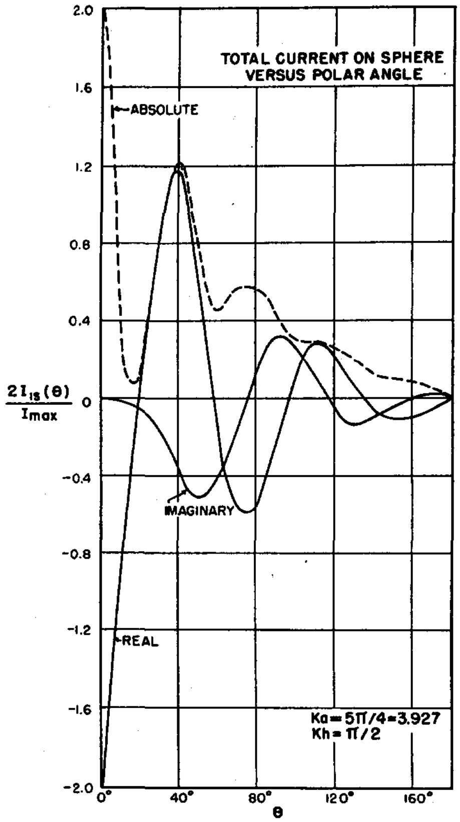

FIG. 7. Total current on sphere *versus* polar angle for $ka=5\pi/4$.

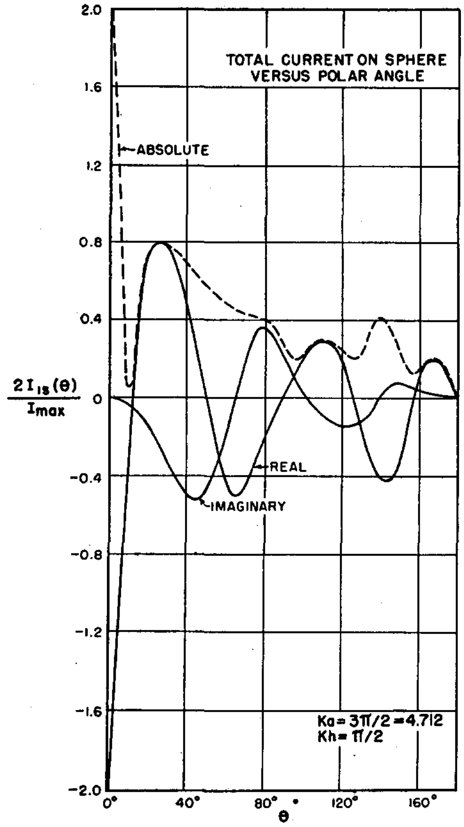

FIG. 8. Total current on sphere *versus* polar angle for $ka=3\pi/2$.

a fair approximation of actual currents in a suitably arranged physical set-up. The determination of such actual distributions using techniques developed in this laboratory for the measurement of magnitude and phase of surface currents is contemplated.

IV. ACKNOWLEDGMENTS

The writers are indebted to Dr. C. T. Tai of this Laboratory for his assistance during the preparation of this manuscript. Computations were made by Miss M. L. Richardson and Mrs. R. Stokey of the laboratory staff.

APPENDIX A

The General Reciprocity Theorem

A given source or distribution of sources maintains a current density i_1 with an associated electromagnetic field E_1, H_1. A second source maintains a current density i_2 with an associated electromagnetic field E_2, H_2. Then, according to the theorem of Lorentz,

$$\nabla\cdot(E_1\times H_2)-\nabla\cdot(E_2\times H_1)=i_1\cdot E_2-i_2\cdot E_1. \quad \text{(A1)}$$

Up to this point there is no relation either between the two sets of fields and currents or the regions in which they are defined. Let it be assumed that the two sets of fields are defined in free space and are bounded by the same geometrical surfaces. The boundary conditions on these surfaces need not be related in any other manner than that they are applied at surfaces that satisfy the same mathematical equations. Let (A1) be multiplied on each side by an element of volume $d\tau$ and integrated over a volume V in empty space that is bounded by a closed surface S and extends to infinity where all fields vanish. Since all volume densities of current vanish in empty space, the right side of (A1) when so integrated vanishes. The left side may be transformed using the divergence theorem, since the fields are continuous in the volume V.

The result is

$$\int_V \{\nabla\cdot(E_1\times H_2-E_2\times H_1)\}d\tau$$
$$=\int_S \{\hat{n}\cdot(E_1\times H_2-E_2\times H_1)\}d\sigma=0, \quad \text{(A2)}$$

where $\hat{n}$ is an external normal to the volume V on the bounding surface S. The only restrictions on the two sets of fields is that they satisfy the field equations, that the volume of integration V be empty space, and that the bounding surface S have the same geometrical shape for both sets but not necessarily the same physical properties.

For periodic time dependence (A2) is valid for complex amplitudes provided both fields vary with the same frequency.

APPENDIX B

Derivation of the Integral Equation

The problem under consideration is to determine the distribution of current on the surface of a perfectly conducting sphere of radius a when excited by a given distribution of current along a linear antenna of very small cross-sectional area. The antenna is placed along the line $\theta=0$ in a system of spherical coordinates r, θ, ϕ with origin at the center of the sphere as shown in Fig. 11. Let the volume V be the entire volume in empty space outside the sphere and the antenna. Similarly, let the inner bounding surface S consist of the surface S_S of the sphere and the surface S_A of the antenna. The outer boundary is at infinity where all fields vanish.

The electromagnetic field vectors E_1 and H_1 are defined to be caused by the actual currents in the antenna and on the surface of the sphere. On the perfectly conducting sphere they must satisfy the boundary conditions:

$$\hat{n}\times E_1=0 \text{ on } S_S, \quad \text{(B1)}$$

$$\hat{n}\times H_1=K_{1S} \text{ on } S_S, \quad \text{(B2)}$$

where $\hat{n}$ is an external normal to the sphere and

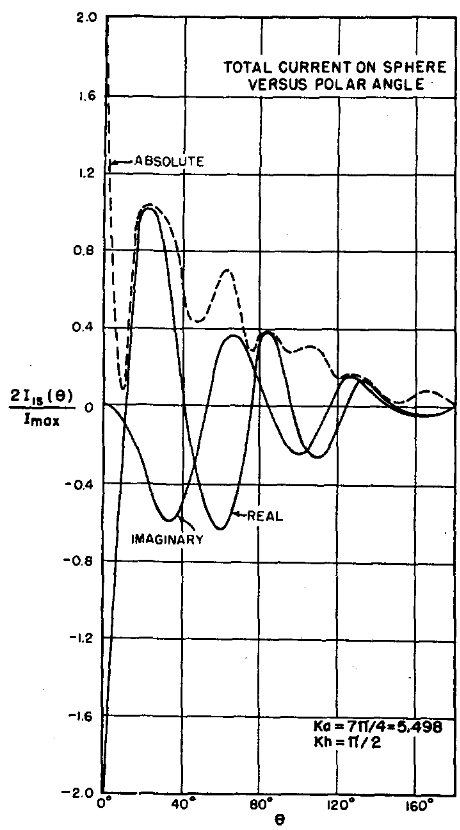

FIG. 9. Total current on sphere *versus* polar angle for $ka=7\pi/4$.

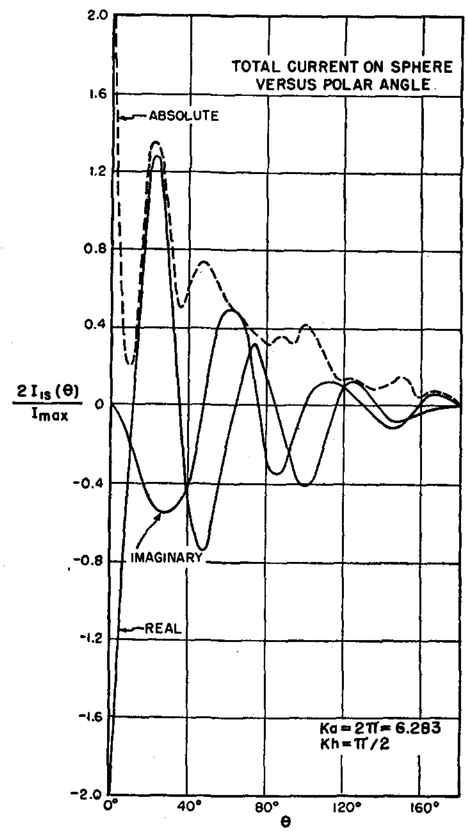

FIG. 10. Total current on sphere *versus* polar angle for $ka=2\pi$.

hence an internal normal to the volume V bounded by S, and where K_{1S} is the surface density of current on the sphere. The current i_{1A} in the antenna and K_{1S} on the sphere are maintained by an appropriate generator or distribution of generators *in the antenna* so disposed that no currents exist in the ϕ-direction (i.e., around an axis through the antenna), in the antenna or on the sphere. Hence,

$$K_{1S}=\hat{\theta}K_{1S}(\theta);\qquad \int_0^{2\pi} K_{1S}(\theta)\cdot a\,\sin\theta d\phi = I_{1S}(\theta), \tag{B3}$$

where $I_{1S}(\theta)$ is the total current crossing a parallel of latitude on the sphere. Also the total current in the antenna is axial, i.e., radial in the system of spherical coordinates. Evidently, complete rotational symmetry is obtained about the axis $\theta=0$ through the antenna and the sphere. The line integral of the tangential component of the magnetic field H_1 around the surface of the antenna is equal to the total axial current $I_{1A}(r)$ in the antenna. Thus, with R_0 the radius of the antenna is

$$\oint H_1\cdot ds=\int_0^{2\pi} H_{1\phi}R_0 d\phi = \int_{\substack{\text{cross}\\ \text{section}}} i_{1A}(r)dS = I_{1A}(r). \tag{B4}$$

The electromagnetic vectors E_2 and H_2 are not actual fields related to the real currents of density i_1 in the antenna or the sphere. They constitute an auxiliary, fictitious field that is introduced to facilitate the derivation and solution of the integral equation. It has been pointed out that E_2 and H_2 must satisfy the field equations and that they must be defined on the same geometrical surfaces, i.e., on the surfaces of the sphere and the antenna, as E_1 and H_1. Similarly, i_2 and E_2 are defined in the same volume as i_1 and E_1. Let it be assumed that the volume V_A (occupied by the antenna for E_1, H_1) is empty space for E_2, H_2. Also, let the volume V_S (occupied by the perfectly conducting sphere for E_1, H_1) be constituted to contain appropriate generators or distributions of generators to maintain the following prescribed field on the surface S_S of the sphere

$$E_{2\phi}=0;\quad E_{2\theta}=U\sin\theta P_n(\cos\theta),\qquad H_{2\theta}=0;\quad H_{2r}=0. \tag{B5}$$

Thus the entire field in empty space and its boundary S_V on the sphere has the components E_{2r}, $E_{2\theta}$, $H_{2\phi}$ which must satisfy the field equations and the prescribed boundary conditions (B5).

After rearrangement of the vector products in (A2) they may be written as follows:

$$\int_{SS}\{H_2\cdot(\hat{n}\times E_1)+E_2\cdot(\hat{n}\times H_1)\} + \int_{SA}\{H_2\cdot(\hat{n}\times E_1)-H_1\cdot(\hat{n}\times E_2)\}d\sigma=0. \tag{B6}$$

Let the integrals in (B6) be considered in turn. As a result of (B1) and (B2) the first integral becomes

$$\int_{SS}\{H_2\cdot(\hat{n}\times E_1)+E_2\cdot(\hat{n}\times H_1)\}d\sigma = \int_{SS} E_2\cdot K_{1S}d\sigma=\int_{SS} E_{2\theta}K_{1S}(\theta)d\sigma = \int_0^{\pi} E_{2\theta}I_{1S}(\theta)a d\theta. \tag{B7}$$

The second and third steps follow from (B3) and (B5).

On the surface S_A of the cylindrical antenna both $(\hat{n}\times E_1)$ and $(\hat{n}\times E_2)$ are constant around each cross section because of rotational symmetry. Furthermore, since $E_\phi=0$,

$$\hat{n}\times E=-\hat{\theta}\times E=-(\hat{\theta}\times\hat{r})E_r-(\hat{\theta}\times\hat{\theta})E_\theta=\hat{\phi}E_r. \tag{B8}$$

Hence, since $H=\hat{\phi}H_\phi$ for both fields,

$$\int_{SA}\{H_2\cdot(\hat{n}\times E_1)-H_1\cdot(\hat{n}\times E_2)\}d\sigma = \int_{r=a}^{r=a+h} E_{1r}dr\int_{\phi=0}^{\phi=2\pi} H_{2\phi}R_0 d\phi - \int_{r=a}^{r=a+h} E_{2r}dr\int_{\phi=0}^{\phi=2\pi} H_{1\phi}R_0 d\phi, \tag{B9}$$

where R_0 is the small radius of the antenna. However, from (B4) the contour integral of the magnetic vector H_1 around the periphery of the antenna equals the total current in the antenna. Accordingly,

$$\int_{r=a}^{a+h} E_{2r}dr \int_{\phi=0}^{2\pi} H_{1\phi}R_0 d\phi = $$

$$= \int_{r=a}^{a+h} E_{2r} I_{1A}(r) dr, \quad \text{(B10)}$$

where $I_{1A}(r)$ is the total current in the antenna. Since the surface S_A (which is the surface of the antenna for E_1H_1) encloses only empty space for E_2H_2, the same contour integral with H_2 instead of H_1 equals only the time rate of change of $\epsilon_0 E_{2r}$ integrated across the small cross-sectional area πR_0^2. Since R_0 can be made as small as required, this quantity can be made negligible so that

$$\int_{r=a}^{r=a+h} E_{1r}dr \int_{\phi=0}^{\phi=2\pi} H_{2\phi}R_0 dr = 0. \quad \text{(B11)}$$

Substitution of (B7), (B10) and (B11) in (B6) gives

$$\int_0^{\pi} E_{2\theta} I_{1S}(\theta) a d\theta = \int_{r=a}^{r=a+h} E_{2r} I_{1A}(r) dr. \quad \text{(B12)}$$

This is the desired integral equation for determining $I_{1S}(\theta)$. $I_{1A}(r)$ is assumed given, $E_{2\theta}$ is the prescribed field given in (B5), and E_{2r} is determined from the vector $E_{2\theta}$ which satisfies the field equations and the boundary conditions (B5).

APPENDIX C

Linear Antenna and Sphere

The solution of the integral equation (B12) for the current $I_{1S}(\theta)$ on a metal sphere on which is erected a linear radiator with current $I_{1A}(r)$ is carried out as follows. Due to the orthogonality of the Legendre polynomials, that is,

$$\int_{-1}^{1} P_m(\mu)P_n(\mu)d\mu = [2/(2n+1)]\delta_{mn}, \quad \text{(C1)}$$

where $\delta_{mn} = \begin{cases} 1 \text{ if } m=n \\ 0 \text{ if } m \neq n \end{cases}$ is the Kronecker delta, it is possible to expand a function in terms of Legendre polynomials. Without limiting the generality the current $I_{1S}(\theta)$ on the surface of the sphere can be represented as follows

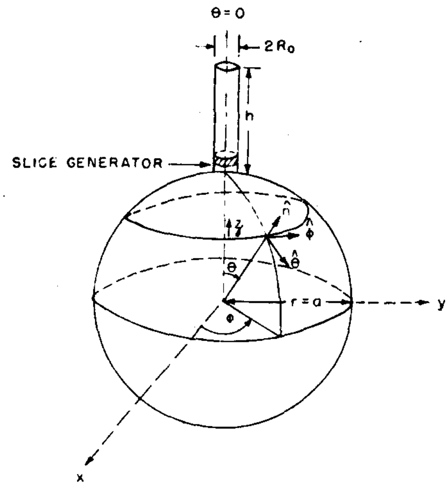

FIG. 11. The geometry of sphere excited by dipole.

$$I_{1S}(\theta) = \sum_{n=0}^{\infty} B_n P_n(\cos\theta). \quad \text{(C2)}$$

B_n are independent of θ and later it will be shown that they are functions of the radius a of the sphere and the height h of the antenna. Physically (C1) means that the total current crossing the parallel of latitude θ is the sum of all the so-called modal currents crossing this parallel of latitude; n is the order of the mode. As n increases, the amplitude of the current in this mode decreases so that it is possible to obtain a rather precise evaluation of $I_{1S}(\theta)$ by summing over the first twelve modes. Mathematically this means that (C1) converges for $0 \leqq \theta \leqq \pi$ rapidly enough so that terms of higher order than twelve contribute very little to the sum.

Expressed in appropriate coordinate form (B12) becomes the following for $r=a$:

$$\int_a^{a+h} E_{2r} I_{1A}(r) dr$$

$$= \int_0^{\pi} E_{2\theta} \sum_{n=0}^{\infty} B_n P_n(\cos\theta) \cdot a d\theta. \quad \text{(C3)}$$

Furthermore, as was previously stated, E_{2r}, $E_{2\theta}$, and $H_{2\phi}$ are the only non-zero components of the field due to currents on the sphere; this means that the field is transverse magnetic TM.

It has been shown[9] that a TM field is uniquely derivable from a function u. Moreover, u satisfies the scalar wave equation:

$$\nabla^2(u/r)+k^2(u/r)=0, \tag{C4}$$

where $k=\omega/c$. The general solution of (C4) is

$$u=\sum_{m=0}^{\infty} A_m P_m(\cos\theta)\rho_m(kr), \tag{C5}$$

where $P_m(\cos\theta)$ is the Legendre polynomial of order m, and

$$\rho_n(x)=(\pi x/2)^{\frac{1}{2}}H_{m+\frac{1}{2}}^{(2)}(x)$$

is the weighted spherical Hankel function of the second kind. The components of the field are obtained from

$$\left.\begin{aligned} E_{2r}&=[k^2+(\partial^2/\partial r^2)]u,\\ E_{2\theta}&=(1/r)(\partial^2/\partial r\partial\theta)u,\\ H_{2\phi}&=-i(k/r)(\partial u/\partial\theta).\end{aligned}\right\} \tag{C6}$$

From (C5) and (C6), $E_{2\theta}$ is evaluated as follows:

$$E_{2\theta}=-(k/a)\sum_{m=0}^{\infty} A_m P_m{}^1(\cos\theta)\rho_m'(ka), \tag{C7}$$

where $P_m{}^1(\cos\theta)$ is the associated Legendre function. In order to evaluate the A_m in (C7), (B5) may be written in the form

$$\begin{aligned} E_{2\theta}&=U\sin\theta P_n(\cos\theta)\\ &=U[[P_{n+1}{}^1(\cos\theta)-P_{n-1}{}^1(\cos\theta)]/(2n+1)]. \end{aligned} \tag{C8}$$

Equating the coefficients of $P_m{}^1(\cos\theta)$ in (C7) and (C8) leads to

$$A_{n+1}=-aU/(2n+1)\cdot k\cdot\rho_{n+1}'(ka), \tag{C9}$$

$$A_{n-1}=+aU/(2n+1)\cdot k\cdot\rho_{n-1}'(ka). \tag{C10}$$

Substitution of these values of A_m in the first of Eqs. (C6) results in the following expression for E_{2r}:

$$E_{2r}=U\left[k^2+\frac{\partial^2}{\partial r^2}\right]\left[\frac{aP_{n-1}(\cos\theta)\rho_{n-1}(kr)}{(2n+1)\cdot k\cdot\rho_{n-1}'(ka)}-\frac{aP_{n+1}(\cos\theta)\rho_{n+1}(kr)}{(2n+1)\cdot k\cdot\rho_{n+1}'(ka)}\right]. \tag{C11}$$

[9] P. Frank and R. v. Mises, *Diff. Gleichungen d. Physik*, (Mary S. Rosenberg, New York) p. 872.

Now let it be assumed that the antenna current is

$$I_{1A}(r)=I_{\max}\sin k(d-r). \tag{C12}$$

With (C12) the left side of (C3) may be computed using (C11):

$$\begin{aligned}\int_a^{a+h} E_{2r}I_{1A}(r)dr&=\frac{UaI_{\max}}{(2n+1)\cdot k\cdot\rho_{n-1}'(ka)}\times\int_a^{a+h}\sin k(d-r)\cdot\left(k^2+\frac{\partial^2}{\partial r^2}\right)\rho_{n+1}(kr)dr\\ &\quad-\frac{UaI_{\max}}{(2n+1)\cdot k\cdot\rho_{n+1}'(ka)}\int_a^{a+h}\sin k(d-r)\cdot\left(k^2+\frac{\partial^2}{\partial r^2}\right)\rho_{n+1}(kr)dr.\end{aligned}$$

This integration is carried out along the antenna, i.e., $\theta=0$, and since $P_n(1)=1$, the Legendre polynomials appearing in (C11) equal unity. Explicitly, the integration yields

$$\begin{aligned}\int_a^{a+h} E_{2r}I_{1A}(r)dr=\frac{UaI_{\max}}{2n+1}&\times\left[\frac{\rho_{n-1}(kd)}{\rho_{n-1}'(ka)}-\cos kh\frac{\rho_{n-1}(ka)}{\rho_{n-1}'(ka)}\right.\\ &\left.-\frac{\rho_{n+1}(kd)}{\rho_{n+1}'(ka)}+\cos kh\frac{\rho_{n+1}(ka)}{\rho_{n+1}'(ka)}\right].\end{aligned} \tag{C13}$$

With (C8) the right-hand side of (C3) becomes:

$$\int_0^{\pi} E_{2\theta}\sum_{n=0}^{\infty} B_nP_n(\cos\theta)\cdot ad\theta=B_n\cdot[U2a/(2n+1)]. \tag{C14}$$

Equating (C13) and (C14), the following expression for B_n is obtained:

$$\begin{aligned}B_n=\frac{I_{\max}}{2}&\left[\frac{\rho_{n-1}(kd)}{\rho_{n-1}'(ka)}-\cos kh\frac{\rho_{n-1}(ka)}{\rho_{n-1}'(ka)}\right.\\ &\left.-\frac{\rho_{n+1}(kd)}{\rho_{n+1}'(ka)}+\cos kh\frac{\rho_{n+1}(ka)}{\rho_{n+1}'(ka)}\right].\end{aligned} \tag{C15}$$

By substituting B_n from (C15) into (C2), $I_{1S}(\theta)$ is determined.